

Package ‘ternvis’

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Type Package

Title Visualisation, Verification and Calibration of Ternary
Probabilistic Forecasts

Version 1.3

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Description A suite of functions for visualising ternary probabilistic forecasts, as discussed in the paper by Jupp (2012) <[doi:10.1098/rsta.2011.0350](https://doi.org/10.1098/rsta.2011.0350)>.

Imports quadprog, maps, dichromat, grid, stats, graphics, grDevices

License GPL-2

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ternvis-package	<i>Visualisation, verification and calibration of ternary forecast data</i>
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Description

A suite of functions for visualising ternary probabilistic forecasts.

Details

Package: ternvis
Type: Package
Version: 1.3
Date: 2025-03-27
License: GPL-2

Colours can be assigned to ternary probabilistic forecasts using [tcolour](#). These colours can be used to produce forecast maps as in the example function [tmap](#). A set of ternary forecasts `p` can be compared with subsequent ternary observations `o` using the function [tverify](#). [plot.tverify](#) then displays this information in a Ternary Reliability Diagram. Calibration is performed using [tgetcal](#) and [tcalibrate](#).

Author(s)

Tim Jupp
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References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.
<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>
<https://arxiv.org/abs/1103.1303/>

See Also

Some concepts adapted from those in package `vcd`.

Examples

```
data(foot)
# see the distribution of forecasts
tplot(foot$p,main="Bookmaker forecasts of \n football matches",
      dimnames=c("Home Win","Draw","Away Win"))
# see how well forecasts compare with results
# create object of class tverify
foot.verify <- tverify(p=foot$p,o=foot$o)

# plot ternary reliability diagram
dev.new()
plot(foot.verify, thresh=3)

# get a (linear) calibration of these data
# create an object of class tverify
foot.calib <- tgetcal(foot.verify)

# plot ternary reliability diagram of calibrated
dev.new()
plot(foot.calib, thresh=3)

data(rain)

tmap(rain,iyr=17,palette=TRUE,circles=FALSE,fac=10)
```

foot

Football betting dataset

Description

A collection of bookmakers' football odds (for home win/draw/away win, expressed as ternary probabilistic forecasts) and subsequent results. The data cover several years of English league games.

Usage

```
data(foot)
```

Format

The quoted odds, transformed to probabilities, are in `foot$p` (a 3-column matrix). The corresponding results of the games are in `foot$o` (a 3-column matrix).

Source

Data taken from www.football-data.co.uk.

Examples

```
data(foot)
# see the distribution of forecasts
tplot(foot$p,main="Bookmaker forecasts of \n football matches",
      dimnames=c("Home Win","Draw","Away Win"))

# see how well forecasts compare with results
# create object of class tverify
foot.verify <- tverify(p=foot$p,o=foot$o)

# plot ternary reliability diagram
dev.new()
plot(foot.verify, thresh=3)

# get a (linear) calibration of these data
# create an object of class tverify
foot.calib <- tgetcal(foot.verify)

# plot ternary reliability diagram of calibrated forecasts
dev.new()
plot(foot.calib, thresh=3)
```

rain

Ternary precipitation forecast data set

Description

An array of monthly seasonal precipitation forecasts and observations over South America, covering 25 years.

Usage

```
data(rain)
```

Format

A list containing 7 objects

- rain\$lats a vector of 35 latitudes
- rain\$lons a vector of 27 longitudes
- rain\$sobs array of ternary observations
- rain\$preds array of ternary forecast probabilities
- rain\$rel array of reliabilities
- rain\$res array of resolutions
- rain\$q array of climatologies

Source

Data from Caio Coelho at <http://eurobrisa.cptec.inpe.br/>.

Examples

```
data(rain)

ilon <- 5 # pick a longitude
ilat <- 6 # pick a latitude
iyr <- 17 # pick a year (in the range 1:25)

rain$lons[ilon]          # the longitude
rain$lat[ilat]           # the latitude
rain$pred[ilon,ilat,iyr,] # the ternary forecast
rain$obs[ilon,ilat,iyr,] # the ternary observation
```

tcalibrate

Calibration of ternary forecasts

Description

Function to apply a calibration to a matrix of ternary forecasts.

Usage

```
tcalibrate(tv, p)
```

Arguments

tv	An object of class <i>tverify</i> , for example produced by tgetcal .
p	An n-by-3 matrix of ternary forecasts which will be calibrated according to the calibration function embedded in <i>dat</i> .

Value

An n-by-3 array of calibrated ternary forecasts.

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, *Philosophical Transactions of the Royal Society*, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

See Also

[tgetcal](#), [tplot](#), [tverify](#)

Examples

```
# get football betting data
data(foot)

# get verification info for these data
tv  <- tverify(p=foot$p,o=foot$o)

# get a linear calibration based on verification info
lc  <- tgetcal(tv)

# consider an uncalibrated forecast p
p   <- cbind(0,0,1)

# see where p is mapped by this calibration
pcal <- tcalibrate(lc,cbind(0,0,1))

# plot the uncalibrated forecast
tplot(p)
# plot the calibrated forecast
tplot(pcal,col="blue",newpage=FALSE,bg="transparent")
```

tcolour

Assign a colour to a ternary forecast

Description

Function to assign a colour to a ternary forecast p , given a ternary climatology q .

Usage

```
tcolour(p = cbind(1, 1, 1)/3, q = cbind(1, 1, 1)/3, m = 0.7,
        flip = FALSE, dich = "none", theta0 = 0)
```

Arguments

p	An n -by-3 matrix of ternary forecasts.
q	A 1-by-3 matrix containing the climatology.
m	Exponent controlling rate of colour saturation.
$flip$	Logical to transpose categories "B" and "A" in barycentric coordinates.
$dich$	Option to render colours as they would appear to a dichromat. Default choice $dich = "none"$ leaves colours unchanged. Choices $dich = "deutan"$ and $dich = "protan"$ show colours as they would appear to those with deutanopia or protanopia (see dichromat).
$theta0$	Angle (degrees) through which to rotate colour palette around climatology q in barycentric coordinates.

Value

A colour, of the format produced by [rgb](#).

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

Examples

```
p <- cbind(0.7,0.2,0.1) # define a forecast
q <- cbind(1,1,1)/3     # define a climatology
pcol <- tcolour(p=p,q=q) # assign a colour
pcol                    # print the colour code to terminal
tplot(p,col=pcol)       # use this colour in a plot
```

tgetcal	<i>Get a linear or quadratic calibration of ternary forecast / observation data</i>
---------	---

Description

Function to obtain a best-fitting calibration to a set of ternary forecasts *p* and corresponding observations *o*.

Usage

```
tgetcal(tv, quad = FALSE)
```

Arguments

tv	An object of the tverify class (as produced by tverify).
quad	A logical describing which functional form to use for calibration. Linear calibration (quad = FALSE) is faster. Quadratic calibration (quad = TRUE) is more sophisticated but can be slow for large data sets.

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

See Also

[tverify](#), [tcalibrate](#)

Examples

```
data(foot)
# see the distribution of forecasts
tplot(foot$p,main="Bookmaker forecasts of \n football matches",
      dimnames=c("Home Win","Draw","Away Win"))

# see how well forecasts compare with results
# create object of class tverify
foot.verify <- tverify(p=foot$p,o=foot$o)

# plot ternary reliability diagram
dev.new()
plot(foot.verify, thresh=3)

# get a (linear) calibration of these data
# create an object of class tverify
foot.calib <- tgetcal(foot.verify)

# plot ternary reliability diagram of calibrated forecasts
dev.new()
plot(foot.calib, thresh=3)
```

tgrid

Create a grid of points in barycentric coordinates

Description

Utility function to generate a grid of ternary forecasts equally spaced in baycentric coordinates.

Usage

```
tgrid(n)
```

Arguments

n An integer describing the number of ‘bins’ along each side of the barycentric triangle.

Value

An n-by-3 matrix of ternary forecasts.

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

See Also

[tplot](#)

Examples

```
tg <- tgrid(6) # make a grid of forecasts in the triangle
tg           # print forecasts to the terminal
tplot(tg)    # plot forecasts
```

tmap

Demonstration of how to make coloured maps of ternary forecasts

Description

An example of how one might make forecast maps using [tpalette](#).

Usage

```
tmap(dat, iyr, circles = FALSE, fac = 10, theta0 = 0,
     dich = "none", m = 0.7, palette = TRUE,
     flip = FALSE)
```

Arguments

dat	An array of forecasts in a suitable format. See <i>Example</i> code below
iyr	In this example, an integer in the range 1 to 25.
circles	circles = FALSE (the default) fills map with coloured polygons. circles = TRUE fills map with circles whose radii are dependent on verification information.
fac	Scaling factor for size of circles when circles = TRUE.

<code>theta0</code>	Angle to rotate colours on palette. See tcolour .
<code>dich</code>	Optionally render colours as they would appear to a dichromat. See tcolour .
<code>m</code>	Parameter controlling rate at which colours saturate in palette. See tcolour .
<code>palette</code>	Logical. If <code>palette = TRUE</code> a colour palette is included.
<code>flip</code>	Logical to choose whether to switch categories <i>B</i> and <i>A</i> . See tcolour .

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

Examples

```
data(rain)

tmap(rain,iyr=17,palette=TRUE,circles=FALSE,fac=10)
```

<code>tpalette</code>	<i>Draw a palette of the colours assigned to ternary forecasts</i>
-----------------------	--

Description

Function to fill barycentric coordinates with a colour palette.

Usage

```
tpalette(q = cbind(1, 1, 1)/3, bars = TRUE, m = 0.7,
         flip = FALSE, lmain = FALSE, theta0 = 0,
         dich = "none", cex = 1, nhist = 12)
```

Arguments

<code>q</code>	A 1-by-3 matrix containing a climatology.
<code>bars</code>	Logical. If "FALSE" a simple colour palette is produced. If "TRUE" barplots representing the forecasts are overlain.
<code>m</code>	Parameter controlling the rate at which colours saturate towards the corners of the triangle.
<code>flip</code>	Logical to transpose categories "B" and "A" in barycentric coordinates.

lmain	Logical. If "TRUE" text is overlaid.
theta0	Angle (degrees) through which to rotate colour palette around climatology q in barycentric coordinates.
dich	Option to render colours as they would appear to a dichromat. Default choice dich = "none" leaves colours unchanged. Choices dich = "deutan" and dich = "protan" show colours as they would appear to those with deutanopia or protanopia (see dichromat).
cex	Factor for expanding the size of the dots when bars is "FALSE".
nhist	Number of barplots along each side of the triangle when bars is "TRUE".

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

See Also

[tmap](#)

Examples

```
dev.new()
tpalette(bars=FALSE) # standard palette

dev.new()
tpalette(bars=TRUE) # palette with histograms overlaid
```

tplot

Plot ternary forecasts in barycentric coordinates.

Description

This function is almost exactly the same as the ternaryplot function in package vcd (on which it is based) except that the ordering of the vertices has been change to bottom left/top/bottom right, and the code has been extended work for an arbitrary scoring matrix L.

Usage

```
tplot(x = cbind(1, 1, 1)/3, L = diag(c(1, 1, 1))/sqrt(2),
      scale = 1, dimnames = NULL,
      dimnames_position = c("corner", "none"),
      dimnames_color = "black", id = NULL, id_color = "black",
      coordinates = FALSE, grid = TRUE, grid_color = "gray",
      labels = c("inside", "outside", "none"),
      labels_color = "darkgray", border = "grey",
      bg = "white", pch = 19, cex = 1, prop_size = FALSE,
      col = "red", main = "ternary plot", sub = NULL,
      newpage = TRUE, pop = TRUE, col.main = "black",
      col.sub = "black", ...)
```

Arguments

<code>x</code>	an n-by-3 matrix of ternary forecasts.
<code>L</code>	A matrix describing the quadratic score function. (Default corresponds to the Brier score).
<code>scale</code>	row sums scale to be used.
<code>dimnames</code>	dimension labels (defaults to the column names of <code>x</code>).
<code>dimnames_position, dimnames_color</code>	position and color of dimension labels.
<code>id</code>	optional labels to be plotted below the plot symbols. <code>coordinates</code> and <code>id</code> are mutually exclusive.
<code>id_color</code>	color of these labels.
<code>coordinates</code>	if TRUE, the coordinates of the points are plotted below them. <code>coordinates</code> and <code>id</code> are mutually exclusive.
<code>grid</code>	if TRUE, a grid is plotted. May optionally be a string indicating the line type (default: "dotted").
<code>grid_color</code>	grid colour.
<code>labels, labels_color</code>	position and colour of the grid labels.
<code>border</code>	colour of the triangle border.
<code>bg</code>	triangle background.
<code>pch</code>	plotting character. Defaults to filled dots.
<code>cex</code>	a numerical value giving the amount by which plotting text and symbols should be scaled relative to the default. Ignored for the symbol size if <code>prop_size</code> is not FALSE.
<code>prop_size</code>	if TRUE, the symbol size is plotted proportional to the row sum of the three variables, i.e., represents the weight of the observation.
<code>col</code>	plotting colour.
<code>main</code>	main title.
<code>sub</code>	a sub title.

newpage if 'TRUE', the plot will appear on a new graphics page.
 pop logical; if 'TRUE', all newly generated viewports are popped after plotting.
 col.main colour for the title.
 col.sub colour for the sub title.
 ... additional graphics parameters (see [par](#)).

Author(s)

Tim Jupp (using code for `vcd:ternaryplot` as source).

Examples

```
tg <- tgrid(6) # make a grid of forecasts in the triangle
tg           # print to terminal
tplot(tg)    # plot them
```

tscale	<i>Rescale a 3-vector to yield a ternary forecast</i>
--------	---

Description

Utility function to rescale 3-vectors to be probability distributions.

Usage

```
tscale(p)
```

Arguments

`p` a vector of length 3.

Details

A utility function.

Value

A vector of length 3 rescaled so that elements are all ≥ 0 and sum to unity.

Author(s)

Tim Jupp

Examples

```
tscale(cbind(1,2,3)) # rescale (1,2,3) to be a ternary forecast.
```

tscore	<i>Calculate the (Brier) score associated with ternary forecasts and observations</i>
--------	---

Description

A utility function.

Usage

```
tscore(p, o, L = diag(c(1, 1, 1))/sqrt(2))
```

Arguments

p	An n-by-3 matrix of ternary forecasts.
o	An n-by-3 matrix of ternary observations.
L	A 3-by-3 matrix defining the quadratic score being used. (Default corresponds to Brier score.)

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

Examples

```
data(foot)
tscore(p=foot$p,o=foot$o)
```

tsetup	<i>Utility function to calculate geometry associated with a quadratic score matrix</i>
--------	--

Description

This is just a useful internal function.

Usage

```
tsetup(L = diag(c(1, 1, 1))/sqrt(2), q = cbind(1, 1, 1)/3)
```

Arguments

L	A 3-by-3 matrix defining the quadratic score being used. (Default corresponds to Brier score.)
q	A 1-by-3 matrix containing the climatology.

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

Examples

```
tsetup() # returns a set of useful objects
```

tverify	<i>Verification of ternary forecasts / observations</i>
---------	---

Description

Function to create verification information associated with a set of forecasts p and associated observations o.

Usage

```
tverify(p, o, q, forceq, ncirc, L)

## Default S3 method:
tverify(p = cbind(1, 1, 1)/3, o = cbind(0, 0, 1),
        q = cbind(1, 1, 1)/3, forceq = FALSE,
        ncirc = 11, L = diag(c(1, 1, 1))/sqrt(2))
## S3 method for class 'tverify'
plot(x, thresh = 0, lsharp = TRUE,
     L = diag(c(1, 1, 1))/sqrt(2),...)
```

Arguments

p	An n-by-3 matrix of ternary forecasts.
o	An n-by-3 matrix of ternary observations.
q	A 1-by-3 matrix containing the climatology.
forceq	A logical. If forceq = TRUE then the input climatology q is used. If forceq = FALSE a climatology is calculated from o.
ncirc	Controls discretisation of barycentric coordinates. (The number of hexagons along each side of the triangle.)
x	An object of the tverify class (as produced by tverify or tgetcal).
thresh	A ‘threshold’ number of forecasts in a ‘bin’, below which the forecast will not be plotted. (Prevents diagram being cluttered by very rare forecasts).
lsharp	Logical controlling whether a ternary sharpness diagram is to be included.
L	A quadratic scoring matrix. The matrix for the Brier score is the default. Full functionality for other scores not yet implemented.
...	Not used.

Value

An object of class `tverify`, which is a list containing the following components:

pbin	array of bins in barycentric coordinates.
Nobs	number of forecasts in each bin.
obar	mean onservation associated with each forecast bin.
score	score associated with each forecast bin.
unc	uncertainty associated with each forecast bin.
rel	reliability associated with each forecast bin.
res	resolution associated with each forecast bin.
scorebar	overall score.
uncbar	overall uncertainty.
relbar	overall reliability.
resbar	overall resolution.

ncirc	number of bins along each side of triangle.
p	input forecast data.
o	input observation data.
assigned	index of bin assigned to each forecast.
L	scoring matrix used.
hexc	array of hexagon centres.
q	climatology.
pk	forecasts set to centre of each bin (<code>tverify</code>) or recalibrated (<code>tgetcal</code>)
ok	mean observations corresponding to forecasts pk
pars	parameters in calibration (<code>tgetcal</code> only).
opt	details of calibration (<code>tgetcal</code> only).
f	calibration function (<code>tgetcal</code> only, for use with <code>tcalibrate</code>).

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

Examples

```
data(foot)           # get football betting data
tv <- tverify(p=foot$p,o=foot$o) # get verification info
plot(tv)             # plot verification info
```

xf	<i>x coordinate of a ternary forecast in barycentric coordinates</i>
----	--

Description

This is a utility function to calculate the x-coordinate of a point in barycentric coordinates. It is unlikely that the user will want to use this function directly.

Usage

```
xf(p = cbind(1, 1, 1)/3, M = tsetup()$M32)
```

Arguments

p	An n-by-3 matrix of ternary forecasts.
M	A 2-by-3 matrix describing the transformation for R^3 to R^2 .

Value

An n-by-1 array of x-values for use in plotting.

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

See Also

[tsetup](#), [yf](#)

Examples

```
xf(p=cbind(.1,.2,.7))
```

yf	<i>y coordinate of a ternary forecast in barycentric coordinates</i>
----	--

Description

This is a utility function to calculate the y-coordinate of a point in barycentric coordinates. It is unlikely that the user will want to use this function directly.

Usage

```
yf(p = cbind(1, 1, 1)/3, M = tsetup())$M32)
```

Arguments

p	An n-by-3 matrix of ternary forecasts.
M	A 2-by-3 matrix describing the transformation for R^3 to R^2 .

Value

An n-by-1 array of y-values for use in plotting.

Author(s)

Tim Jupp

References

Jupp TE, Lowe R, Stephenson DB, Coelho CAS (2012) On the visualization, verification and recalibration of ternary probabilistic forecasts, Philosophical Transactions of the Royal Society, volume 370, pages 1100-1120.

<https://rsta.royalsocietypublishing.org/content/370/1962/1100.full/>

<https://arxiv.org/abs/1103.1303/>

See Also

[tsetup, xf](#)

Examples

```
yf(p=cbind(.1,.2,.7))
```

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